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COMMENTS AND REPLIES

Comment on 'Monte Carlo simulation study of the two-stage percolation transition in enhanced binary trees'

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Abstract

The enhanced binary tree (EBT) is a nontransitive graph which has two percolation thresholds p_{c1} and p_{c2} with $p_{c1} < p_{c2}$. Our Monte Carlo study implies that the second threshold p_{c2} is significantly lower than a recent claim by Nogawa and Hasegawa (2009 *J. Phys. A: Math. Theor.* **42** 145001). This means that p_{c2} for the EBT does not obey the duality relation for the thresholds of dual graphs $p_{c2} + \overline{p}_{c1} = 1$ which is a property of a transitive, nonamenable, planar graph with one end. As in regular hyperbolic lattices, this relation instead becomes an inequality $p_{c2} + \overline{p}_{c1} < 1$. We also find that the critical behavior is well described by the scaling form previously found for regular hyperbolic lattices.

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Recently, Nogawa and Hasegawa [1] have reported the two-stage percolation transition on a nonamenable graph which they called the enhanced binary tree (EBT). While the first transition had little ambiguity, they mentioned that the behavior at the second threshold did not look like a usual continuous phase transition.

A quantity of interest was the mass of the root cluster, denoted as s_0, where the root cluster was defined as the one including the root node of the EBT. Using this observable, we briefly check the first transition point, p_{c1}, where an unbounded cluster begins to form. As in [2], we have used the Newman–Ziff algorithm [3, 4] and taken averages over 10^6 samples throughout this work. The number of generations, L, of the EBT defines a typical length scale of the system, and [1] showed the finite-size scaling of s_0 as

$$s_0/L \propto \tilde{f}_1[(p - p_{c1})L^{1/\nu}], \tag{1}$$

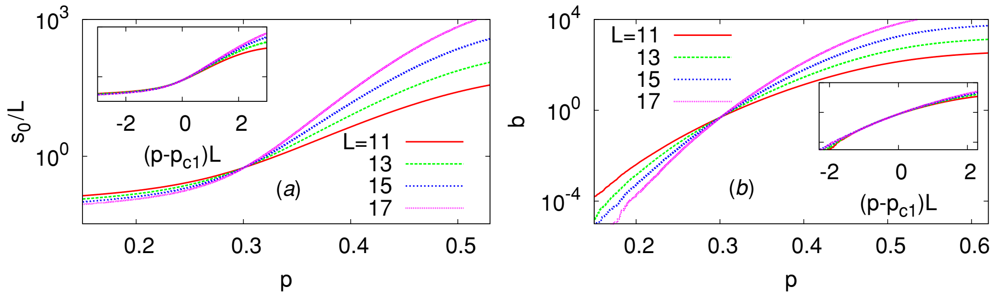

Figure 1. (a) Mass of the root cluster, s_0, divided by L, the number of generations in the EBT. The crossing point indicates the first percolation transition point, p_{c1}, where an unbounded cluster emerges. Inset: Scaling collapse by equation (1) with $p_{c1} = 0.304$ found in [1]. (b) The number of boundary points connected to the root node, denoted as b, also shows a crossing point at $p = p_{c1}$. Inset: Scaling collapse by equation (2) with the same p_{c1} as above.

with $\nu = 1$. Figure 1(a) confirms both the percolation threshold p_{c1} and the scaling form, equation (1). Equivalently, one can measure b, the number of boundary points connected to the root node, which becomes finite above p_{c1} as shown in figure 1(b). It also scales as

$$b \propto \tilde{f}_2[(p - p_{c1})L^{1/\nu}], \tag{2}$$

with the same exponent ν. Comparing this with [2], we see that the percolation transition in the EBT at $p = p_{c1}$ belongs to the same universality class as that of regular hyperbolic lattices. One may argue that this scaling form actually corresponds to the case of Cayley trees [2]. The convincing results in figure 1 imply that the estimation in [1] for the dual of the EBT, $\overline{p}_{c1} = 0.436$, is also correct.

On the other hand, the second percolation transition at $p = p_{c2}$ indicates uniqueness of the unbounded cluster. We have thus employed a direct observable to detect this transition, i.e., the ratio between the first and second largest cluster masses [2]. The idea is that even the second largest cluster would become negligible if there can exist only one unique unbounded cluster. Measuring s_2/s_1 in the EBT, where s_i means the ith largest cluster mass, we have found the second transition at $p_{c2} \approx 0.48$ (figure 2(a)), certainly lower than Nogawa and Hasegawa's estimation, $p = 0.564$.

As an alternative quantity for p_{c2}, we divide b by the number of all the boundary points, B. This fraction b/B is supposed to become finite above p_{c2} [2]. Based on the Cayley tree result [2], we have assumed that as the system size N varies, one can write down the following asymptotic form:

$$b/B \sim c_1 N^{\phi-1} + c_2, \tag{3}$$

with some constants c_1 and c_2 and an exponent ϕ. From the finite-size data, we extrapolate the large-system limit by equation (3), which suggests $p_{c2} \approx 0.49$ (figure 2(b)). This is very close to the estimation above from s_2/s_1. Moreover, in accordance with equation (3), we have suggested the following scaling hypothesis to describe the critical behavior at this transition point [2]:

$$b/B \propto N^{\phi-1} \tilde{f}_3[(p - p_{c2})N^{1/\bar{\nu}}], \tag{4}$$

with an exponent $\bar{\nu}$. Applying this hypothesis to EBT data, we see that $\phi = 0.84$ and $1/\bar{\nu} = 0.12$ give a good fit (figure 2(c)) with the same value of $p_{c2} = 0.48$, where the numeric values of the scaling exponents are again consistent with [2].

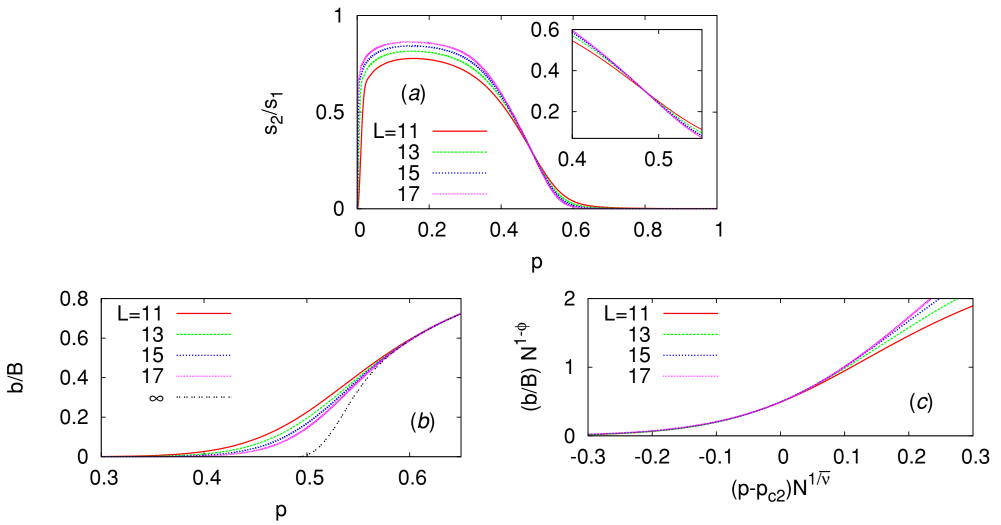

Figure 2. (*a*) Ratio between the second largest cluster mass, s_2, and the first largest one, s_1. The crossing point lies at $p \approx 0.48$, which implies that only one cluster will remain dominant in the system. (*b*) The number of boundary points connected to the root node, b, divided by the total number of boundary points, B. The dotted black curve marked by ∞ indicates the extrapolation result from equation (3). (*c*) Scaling collapse according to equation (4), where we set $p_{c2} = 0.48$, $\phi = 0.84$ and $1/\bar{\nu} = 0.12$.

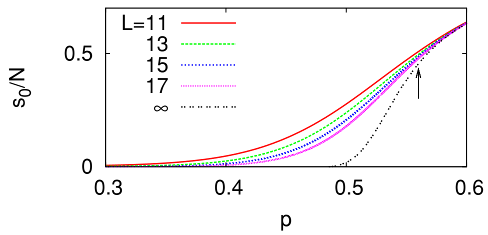

Figure 3. Mass fraction of the root cluster. The extrapolation result is represented by the dotted black curve named as ∞. The arrow indicates $p = 0.564$, predicted as the transition point in [1].

To make a direct comparison to the observation in [1], we have also calculated the mass fraction of the root cluster, s_0/N, as a function of p. As above, performing extrapolation to the large system limit, we see that this quantity becomes positive finite at $p \gtrsim 0.49$ (figure 3).

All of these observations suggest that the predicted value of p_{c2} in [1] is too high, and it seems that this overestimation led them to consider 'discontinuity' since s_0/N became already so large at that point as shown in figure 3.

Finally, even though our estimation suggests such a different p_{c2} that $p_{c2} + \overline{p}_{c1} < 1$, we note that it does not violate the duality relation proved in [5] for a transitive, nonamenable, planar graph with one end: as Nogawa and Hasegawa correctly pointed out [1], the EBT does not possess transitivity. The inequality $p_{c2} + \overline{p}_{c1} < 1$ was explicitly verified for a pair of hyperbolic dual lattices {7, 3} and {3, 7} in [2]. This inequality means the existence of a narrow region of p between p_{c2} and $1 - \overline{p}_{c1}$, where one would find a unique unbounded cluster in a given graph whereas infinitely many unbounded clusters in its dual graph. Such a region

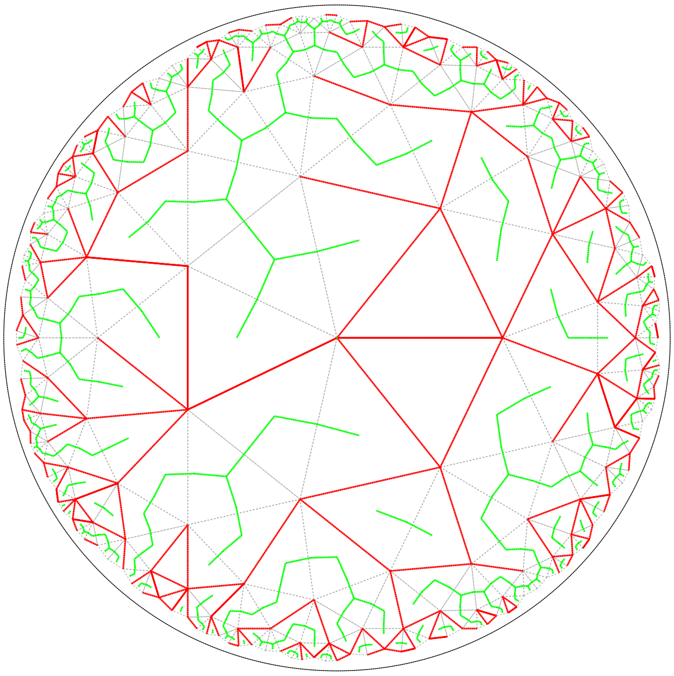

Figure 4. Visualization of a triangular hyperbolic lattice projected on the Poincaré disk, where the maximum length from the origin is chosen to be $L = 4$. Bonds are randomly occupied with a probability of $p = 0.42$, which are colored red, while only the rest of them appear as occupied in the dual lattice, as colored green, so that the dual probability corresponds to $\overline{p} = 1 - p = 0.58$. Note that p lies between p_{c2} and $1 - \overline{p}_{c1}$, since this structure has $p_{c2} \approx 0.37$ and its dual has $\overline{p}_{c1} \approx 0.53$, according to [2]. While most clusters have already been absorbed into the largest red cluster, many of green clusters still have radii comparable to L since $\overline{p}_{c1} < \overline{p} < \overline{p}_{c2} \approx 0.72$.

does not exist for a transitive case [5]. A typical state in this region is illustrated in figure 4, which shows a situation with many unbounded clusters of radii comparable to L at the same time as a single unbounded cluster occupies the dominant part of the dual graph.

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